Planes in $\mathbb{R}^3$
We described a line as the set of position vectors expressible as $r_0 + v$, where $r_0$ was a position vector of a point in $L$ and $v$ was any vector parallel to $L$.

We can describe a plane the same way: the set of position vectors expressible as the sum of a position vector to a point in $P$ and an arbitrary vector parallel to $P$.

Choose a vector $n$ which is orthogonal to the plane and choose an arbitrary point $P_0$ in the plane.

How can we use this data to describe all the other points $P$ which lie in the plane?
Let $r_0$ and $r$ be the position vectors of $P_0$ and $P$ respectively. The normal vector $n$ is orthogonal to every vector in the plane. In particular $n$ is orthogonal to $r - r_0$ and so we have
\[ n \cdot (r - r_0) = 0. \]

This equation
\[ n \cdot (r - r_0) = 0. \quad (1) \]
can be rewritten as
\[ n \cdot r = n \cdot r_0. \quad (2) \]
Either of the equations (1) or (2) is called a vector equation of the plane.
Example 1
Find a vector equation for the plane passing through \( P_0 = (0, -2, 3) \) and normal to the vector \( n = 4i + 2j - 3k \).

We have \( r_0 = (0, -2, 3) \) and \( n = (4, 2, -3) \). Thus the vector form is
\[
n \cdot (r - r_0) = 0,
\]
or
\[
(4i + 2j - 3k) \cdot [(x - 0)i + (y + 2)j + (z - 3)k] = 0.
\]
Expanding this gives us a scalar equation for the plane...

Given \( n = \langle A, B, C \rangle \), \( r = \langle x, y, z \rangle \) and \( r_0 = \langle x_0, y_0, z_0 \rangle \), the vector equation \( n \cdot (r - r_0) = 0 \) becomes
\[
\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,
\]
or
\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{3}
\]
Equation (3) is the scalar equation of the plane through \( P_0(x_0, y_0, z_0) \) with normal vector \( n = \langle A, B, C \rangle \).

The equation
\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.
\]
can be written more simply in standard form
\[
Ax + By + Cz + D = 0,
\]
where \( D = -(Ax_0 + By_0 + Cz_0) \).

If \( D = 0 \), the plane passes through the origin.
Example 2
Find a scalar equation for the plane passing through $P_0 = (0, -2, 3)$ and normal to the vector $n = 4i + 2j - 3k$.

The vector form is

$$(4i + 2j - 3k) \cdot [(x - 0)i + (y + 2)j + (z - 3)k] = 0,$$

which in scalar form becomes

$$4(x - 0) + 2(y + 2) - 3(z - 3) = 0,$$

and this is equivalent to

$$4x + 2y - 3z = -13.$$
The first step in this example was finding the normal vector $\mathbf{n}$, but in fact, there’s another way to do this.

Recall that in $\mathbb{R}^3$ only, there is a product of two vectors called a cross product. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector denoted $\mathbf{a} \times \mathbf{b}$ which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. If we have two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ parallel to our plane, then $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is a normal vector.

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**Example 4**

Consider the two planes

$$x - y + z = -1 \quad \text{and} \quad 2x + y + 3z = 4.$$  

Explain why the planes above are not parallel and find a direction vector for the line of intersection.

Two planes are parallel if and only if their normal vectors are parallel. Normal vectors for the two planes above are for example

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

respectively. These vectors are not parallel, so the planes can’t be parallel and must intersect. A vector $\mathbf{v}$ parallel to the line of intersection is a vector which is orthogonal to both the normal vectors above. We can find such a vector by calculating the cross product of the normal vectors:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -4\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$  

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**Example 5**

Find the line through the origin and parallel to the line of intersection of the two planes

$$x + 2y - z = 2 \quad \text{and} \quad 2x - y + 4z = 5.$$  

The planes have respective normals

$$\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}.$$  

A direction vector for their line of intersection is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$  

A vector parametric equation of the line is

$$\mathbf{r} = t(7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}),$$  

since the line passes through the origin.
Parametric equations for this line are, for example,

\[ x = 7t \]
\[ y = -6t \]
\[ z = -5t \]

and the corresponding symmetric equations are

\[ \frac{x}{7} = \frac{y}{-6} = \frac{z}{-5}. \]

Recommended exercises for review

Stewart §10.5: 1, 3, 15, 19, 25, 29, 35

Overview

Yesterday we introduced equations to describe lines and planes in \( \mathbb{R}^3 \):

- \( \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \)
  - The vector equation for a line describes arbitrary points \( \mathbf{r} \) in terms of a specific point \( \mathbf{r}_0 \) and the direction vector \( \mathbf{v} \).

- \( \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \)
  - The vector equation for a plane describes arbitrary points \( \mathbf{r} \) in terms of a specific point \( \mathbf{r}_0 \) and the normal vector \( \mathbf{n} \).

Question

How can we find the distance between a point and a plane in \( \mathbb{R}^3 \)? Between two lines in \( \mathbb{R}^3 \)? Between two planes? Between a plane and a line?

(From Stewart §10.5)
Distances in \( \mathbb{R}^3 \)

The distance between two points is the length of the line segment connecting them. However, there’s more than one line segment from a point \( P \) to a line \( L \), so what do we mean by the distance between them?

The distance between any two subsets \( A, B \) of \( \mathbb{R}^3 \) is the smallest distance between points \( a \) and \( b \), where \( a \) is in \( A \) and \( b \) is in \( B \).

- To determine the distance between a point \( P \) and a line \( L \), we need to find the point \( Q \) on \( L \) which is closest to \( P \), and then measure the length of the line segment \( PQ \). This line segment is orthogonal to \( L \).
- To determine the distance between a point \( P \) and a plane \( S \), we need to find the point \( Q \) on \( S \) which is closest to \( P \), and then measure the length of the line segment \( PQ \). Again, this line segment is orthogonal to \( S \).

In both cases, the key to computing these distances is drawing a picture and using one of the vector product identities.

Distance from a point to a plane

We find a formula for the distance \( s \) from a point \( P_1 = (x_1, y_1, z_1) \) to the plane \( Ax + By + Cz + D = 0 \).

Let \( P_0 = (x_0, y_0, z_0) \) be any point in the given plane and let \( b \) be the vector corresponding to \( P_0P_1 \). Then

\[
b = (x_1 - x_0, y_1 - y_0, z_1 - z_0).
\]

The distance \( s \) from \( P_1 \) to the plane is equal to the absolute value of the scalar projection of \( b \) onto the normal vector \( n = (A, B, C) \).

\[
s = \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}.
\]

Since \( P_0 \) is on the plane, its coordinates satisfy the equation of the plane and so we have \( Ax_0 + By_0 + Cz_0 + D = 0 \). Thus the formula for \( s \) can be written

\[
s = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.
\]
Example 6
We find the distance from the point $(1, 2, 0)$ to the plane
$3x - 4y - 5z - 2 = 0$.

From the result above, the distance $s$ is given by

$$s = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

where $(x_0, y_0, z_0) = (1, 2, 0)$,

$A = 3, B = -4, C = -5$ and $D = -2$.

This gives

$$s = \frac{|3\cdot 1 + (-4)\cdot 2 + (-5)\cdot 0 - 2|}{\sqrt{3^2 + (-4)^2 + (-5)^2}}$$

$$= \frac{7}{\sqrt{50}} = \frac{7\sqrt{2}}{10}.$$ 

Distance from a point to a line

Question

Given a point $P_0 = (x_0, y_0, z_0)$ and a line $L$ in $\mathbb{R}^3$, what is the distance from $P_0$ to $L$?

Tools:
- describe $L$ using vectors
- $||u \times v|| = ||u|| ||v|| \sin \theta$

Distance from a point to a line

Let $P_0 = (x_0, y_0, z_0)$ and let $L$ be the line through $P_1$ and parallel to the nonzero vector $v$. Let $r_0$ and $r_1$ be the position vectors of $P_0$ and $P_1$ respectively. $P_2$ on $L$ is the point closest to $P_0$ if and only if the vector $\overrightarrow{P_2P_0}$ is perpendicular to $L$.

The distance from $P_0$ to $L$ is given by

$$s = ||\overrightarrow{P_2P_0}|| = ||\overrightarrow{P_1P_0}|| \sin \theta = ||r_0 - r_1|| \sin \theta$$

where $\theta$ is the angle between $r_0 - r_1$ and $v$. 

Since
\[ \|(r_0 - r_1) \times v\| = ||r_0 - r_1|| \|v\| \sin \theta \]
we get the formula
\[ s = ||r_0 - r_1|| \sin \theta = \frac{||(r_0 - r_1) \times v\|}{\|v\|} \]

Example 7
Find the distance from the point \((1,1,-1)\) to the line of intersection of the planes
\[ x + y + z = 1, \quad 2x - y - 5z = 1. \]

The direction of the line is given by \(v = n_1 \times n_2\) where \(n_1 = i + j + k\), and \(n_2 = 2i - j - 5k\).

\[ v = n_1 \times n_2 = -4i + 7j - 3k. \]

In the diagram, \(P_1\) is an arbitrary point on the line. To find such a point, put \(x = 1\) in the first equation. This gives \(y = -z\) which can be used in the second equation to find \(z = 1/4\), and hence \(y = -1/4\).

Here \(\overrightarrow{P_1P_0} = r_0 - r_1 = \frac{5}{2}j - \frac{5}{2}k\). So
\[ s = \frac{||(r_0 - r_1) \times v\|}{\|v\|} = \frac{||\left(\frac{5}{2}j - \frac{5}{2}k\right) \times (-4i + 7j - 3k)||}{\sqrt{(-4)^2 + 7^2 + (-3)^2}} = \frac{||5i + 5j + 5k||}{\sqrt{74}} = \sqrt{\frac{75}{74}}. \]
Distance between two lines

Let \( L_1 \) and \( L_2 \) be two lines in \( \mathbb{R}^3 \) such that
- \( L_1 \) passes through the point \( P_1 \) and is parallel to the vector \( v_1 \)
- \( L_2 \) passes through the point \( P_2 \) and is parallel to the vector \( v_2 \).
Let \( r_1 \) and \( r_2 \) be the position vectors of \( P_1 \) and \( P_2 \) respectively.
Then parametric equation for these lines are
\[
L_1 \quad r = r_1 + tv_1 \\
L_2 \quad \tilde{r} = r_2 + sv_2
\]
Note that \( r_2 - r_1 = \overrightarrow{P_1P_2} \).
We want to compute the smallest distance \( d \) (simply called the distance) between the two lines.
If the two lines intersect, then \( d = 0 \). If the two lines do not intersect we can distinguish two cases.

**Case 1:** \( L_1 \) and \( L_2 \) are parallel and do not intersect.
In this case the distance \( d \) is simply the distance from the point \( P_2 \) to the line \( L_1 \) and is given by
\[
d = \frac{||\overrightarrow{P_1P_2} \times v_1||}{||v_1||} = \frac{||(r_2 - r_1) \times v_1||}{||v_1||}
\]

**Case 2:** \( L_1 \) and \( L_2 \) are skew lines.
If \( P_3 \) and \( P_4 \) (with position vectors \( r_3 \) and \( r_4 \) respectively) are the points on \( L_1 \) and \( L_2 \) that are closest to one another, then the vector \( \overrightarrow{P_3P_4} \) is perpendicular to both lines (i.e. to both \( v_1 \) and \( v_2 \)) and therefore parallel to \( v_1 \times v_2 \). The distance \( d \) is the length of \( \overrightarrow{P_3P_4} \).
Notice that \( d = ||r_4 - r_3|| \), which we can rewrite as
\[
d = \frac{||(r_4 - r_3) \cdot (v_1 \times v_2)||}{||v_1 \times v_2||}
\]
because \( r_4 - r_3 \) is parallel to \( v_1 \times v_2 \).
What’s the point of doing this? Of course we don’t know what $r_4$ or $r_3$ is. Here’s the trick: Notice that

$$r_4 = r_2 + tv_2 \quad r_3 = r_1 + sv_1$$

for some $s$ and $t$.

Now substitute these into our dimension formula, obtaining

$$d = \frac{|(r_2 - r_1 + tv_2 - sv_1) \cdot (v_1 \times v_2)|}{||v_1 \times v_2||}$$

which simplifies, since $v_1 \times v_2$ is orthogonal to both $v_1$ and $v_1$, to

$$d = \frac{|(r_2 - r_1) \cdot (v_1 \times v_2)|}{||v_1 \times v_2||}$$

Thus we don’t need to know $r_4$ or $r_3$ explicitly at all! (Exercise — find formulas for them!)

**Example 8**

Find the distance between the skew lines

$$\begin{align*}
x + 2y &= 3 \\
y + 2z &= 3 \quad \text{and} \quad x + y + z &= 6 \\
x - 2z &= -5
\end{align*}$$

We can take $P_1 = (1, 1, 1)$, a point on the first line, and $P_2 = (1, 2, 3)$ a point on the second line. This gives $r_2 - r_1 = j + 2k$. 
Now we need to find $v_1$ and $v_2$:

$$v_1 = (i + 2j) \times (j + 2k) = 4i - 2j + k,$$

and

$$v_2 = (i + j + k) \times (i - 2k) = -2i + 3j - k.$$

This gives

$$v_1 \times v_2 = -i + 2j + 8k.$$

The required distance $d$ is the length of the projection of $r_2 - r_1$ in the direction of $v_1 \times v_2$, and is given by

$$d = \frac{|(r_2 - r_1) \cdot (v_1 \times v_2)|}{|v_1 \times v_2|},$$

$$= \frac{|(j + 2k) \cdot (-i + 2j + 8k)|}{\sqrt{(-1)^2 + 2^2 + 8^2}},$$

$$= \frac{18}{\sqrt{69}}.$$