Let $F : M_{2 \times 2} \to \mathbb{P}_2$ be a linear transformation given by

$$F \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + b + c + (a - b)x + (d - c)x^2$$

Note that the **kernel** of this transformation will be a $2 \times 2$ matrix. It is the set of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy

$$a + b + c = 0$$
$$a - b = 0$$
$$d - c = 0$$

To find these matrices we solve the system of homogeneous equation given by

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}
\\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & -2 & -1 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1/2 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1/2 \\
0 & 1 & 0 & 1/2 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

This gives \( a = -\frac{1}{2}d \), \( b = -\frac{1}{2}d \) and \( c = d \) so that the matrices we are looking for are of the form
\[
\begin{bmatrix}
-\frac{1}{2}d & -\frac{1}{2}d \\
\frac{1}{2}d & \frac{1}{2}d \\
\end{bmatrix} = d \begin{bmatrix}
-1/2 & -1/2 \\
1 & 1
\end{bmatrix}
\]

So any matrix that is a scalar multiple of
\[
\begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\] is in the kernel of \( F \).
The **range** of the transformation is possibly harder to find, though not impossible. The range is a subset of the polynomials of degree at most 2, \( \mathbb{P}_2 \). What we want to know is whether it is all of \( \mathbb{P}_2 \) or only part of it.

Essentially we want to know, if we are given a polynomial \( a_0 + a_1x + a_2x^2 \) can we always find \( a, b, c, d \) such that

\[
F \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a_0 + a_1x + a_2x^2.
\]

This means we need to be able to solve the equations:

\[
\begin{align*}
    a + b + c &= a_0 \\
    a - b &= a_1 \\
    d - c &= a_2
\end{align*}
\]

and that means row reducing the matrix

\[
\begin{bmatrix}
    1 & 1 & 1 & 0 & a_0 \\
    1 & -1 & 0 & 0 & a_1 \\
    0 & 0 & -1 & 1 & a_2
\end{bmatrix}
\]
We don’t need to do as many steps as we did before to show that this will always have a solution. It is sufficient just to make the first step

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & a_0 \\
1 & -1 & 0 & 0 & a_1 \\
0 & 0 & -1 & 1 & a_2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 0 & a_0 \\
0 & -2 & -1 & 0 & a_1 - a_0 \\
0 & 0 & -1 & 1 & a_2
\end{bmatrix}
\]

The matrix is now in echelon form and we know it will always have a solution. This means that the range of $F$ is all of $\mathbb{P}_2$. 
We consider the linear transformation $H : \mathbb{P}_2 \rightarrow M_{2 \times 2}$ given by

$$H(a + bx + cx^2) = \begin{bmatrix} a + b & a - b \\ c & c - a \end{bmatrix}$$

The kernel of $H$ is a subset of $\mathbb{P}_2$, the polynomials of degree at most 2, and is the set of polynomials $a + bx + cx^2$ with

$$a + b = 0$$
$$a - b = 0$$
$$c = 0$$
$$-a + c = 0$$

It is easy to see that the only solution to this set of equations is $a = b = c = 0$, so the kernel of $H$ is just the zero polynomial.
The **range** of $H$ is a subset of the $2 \times 2$ matrices, and again we want to know if it is all of $M_{2 \times 2}$ or only part of it. So we want to know if we are given any matrix

$$
\begin{bmatrix}
x & y \\
z & t
\end{bmatrix}
$$

can we always find $a$, $b$ and $c$ such that

$$
H(a + bx + cx^2) = \begin{bmatrix} x & y \\ z & t \end{bmatrix}.
$$

To find out we need to solve the equations

\begin{align*}
a + b &= x \\
a - b &= y \\
c &= z \\
-a + c &= t
\end{align*}
This gives an augmented matrix

\[
\begin{bmatrix}
1 & 1 & 0 & x \\
1 & -1 & 0 & y \\
0 & 0 & 1 & z \\
-1 & 0 & 1 & t \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & x \\
0 & -2 & 0 & y - x \\
0 & 0 & 1 & z \\
0 & 1 & 1 & t + x \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & x \\
0 & 1 & 1 & t + x \\
0 & -2 & 0 & y - x \\
0 & 0 & 1 & z \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & x \\
0 & 1 & 1 & t + x \\
0 & -2 & 0 & y - x \\
0 & 0 & 1 & z \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & x \\
0 & 1 & 1 & t + x \\
0 & 0 & 2 & y + x + 2t \\
0 & 0 & 1 & z \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & x \\
0 & 1 & 1 & t + x \\
0 & 0 & 0 & y + x + 2t - 2z \\
0 & 0 & 1 & z \\
\end{bmatrix}
\]

The third row shows that we only have a solution to this set of equations when
\(y + x + 2t - 2z = 0\). This means that the range of \(H\) is the set of all matrices
\[
\begin{bmatrix}
x \\
y \\
z \\
t \\
\end{bmatrix}
\]
where \(x, y, z, t\) satisfy \(y + x + 2t - 2z = 0\).
Consider $T$ the linear transformation $T : M_{2 \times 2} \to M_{2 \times 2}$ given by
\[ T(A) = A + A^T \]
where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

More explicitly
\[ T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b + c \\ b + c & 2d \end{bmatrix} \]

The **kernel** of $T$ is the set of all matrices for which
\[ T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

For this to happen we require that $a = d = 0$, $c = -b$, so that the kernel of $T$ is the set of matrices of the form
\[ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]
Finding the **range** of $T$ is not so difficult in this case. We can see immediately from

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 2a & b + c \\ b + c & 2d \end{bmatrix}$$

that the effect of $T$ on any matrix is to produce a symmetric matrix (a matrix where $A = A^T$). Furthermore any symmetric matrix can be made by the appropriate choice of $a, b, c, d$. Thus the range of $T$ is the set of symmetric $2 \times 2$ matrices.