MATH1014
Semester 2
Administrative Overview

Lecturers:

Stephen Roberts  Qian Wang
linear algebra  calculus
stephen.roberts@anu.edu.au  email TBA

Assessment

- Midsemester exam (date TBA, request for week 7) (20%)
- Final exam (50%)
- Web Assign quizzes (lowest 2 dropped) (10%)
- Workshop quizzes (lowest 2 dropped) (8%)
- Workshop participation (lowest 2 dropped) (2%)
- Two Written assignments (due weeks 6 and 10) (10%)

Tips for success:
- Ask questions!
- Make use of the available resources!
- Don’t fall behind!

Linear Algebra

- We will be covering most of the material in Stewart, Sections 10.1, 10.2, 10.3 and 10.4, and Lay Chapters 4 and 5, and Chapter 6, Sections 1 - 6.
- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, dot products, cross products in $\mathbb{R}^3$, planes and lines in $\mathbb{R}^3$ (Stewart).
- Properties of Vector Spaces and Subspaces.
- Linear Independence, bases and dimension, change of basis.
- Applications to difference equations, Markov chains.
- Eigenvalues and eigenvectors.
- Orthogonality, Gram-Schmidt process. Least squares problem.
Coordinates, Vectors and Geometry in $\mathbb{R}^3$

From Stewart, §10.1, §10.2

Question: How do we describe 3-dimensional space?

- Coordinates
- Lines, planes, and spheres in $\mathbb{R}^3$
- Vectors

Euclidean Space and Coordinate Systems

We identify points in the plane ($\mathbb{R}^2$) and in three-dimensional space ($\mathbb{R}^3$) using coordinates.

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

reads as “$\mathbb{R}^3$ is the set of ordered triples of real numbers”.

We first choose a fixed point $O = (0, 0, 0)$, called the origin, and three directed lines through $O$ that are perpendicular to each other. We call these the coordinate axes and label them the $x$-axis, the $y$-axis and the $z$-axis.

Usually we think of the $x$- and $y$-axes as being horizontal and the $z$-axis as being vertical.

Together, $\{x, y, z\}$ form a right-handed coordinate system.

Compare this to the axes we use to describe $\mathbb{R}^2$, where the $x$-axis is horizontal and the $y$-axis is vertical.
The Distance Formula

**Definition**
The distance \( |P_1P_2| \) between the points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) is

\[
|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**Definition**
The distance \( |P_1P_2| \) between the points \( P_1 = (x_1, y_1, z_1) \) and \( P_2 = (x_2, y_2, z_2) \) is

\[
|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

### 1.1 Surfaces in \( \mathbb{R}^3 \)

Lines, planes, and spheres are special sets of points in \( \mathbb{R}^3 \) which can be described using coordinates.

**Example 1**
The sphere of radius \( r \) with centre \( C = (c_1, c_2, c_3) \) is the set of all points in \( \mathbb{R}^3 \) with distance \( r \) from \( C \):

\[
S = \{ P : |PC| = r \}.
\]

Equivalently, the sphere consists of all the solutions to this equation:

\[
(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = r^2.
\]

**Example 2**
The equation \( z = -5 \) in \( \mathbb{R}^3 \) represents the set \( \{ (x, y, z) | z = -5 \} \), which is the set of all points whose \( z \)-coordinate is \(-5\). This is a horizontal plane that is parallel to the \( xy \)-plane and five units below it.
Example 3
What does the pair of equations $y = 3, z = 5$ represent? In other words, describe the set of points
\[
\{(x, y, z) : y = 3 \text{ and } z = 5\} = \{(x, 3, 5)\}.
\]

Connections with linear equations
Recall from 1013 that a system of linear equations defines a solution set. When we think about the unknowns as coordinate variables, we can ask what the solution set looks like.

* A single linear equation with 3 unknowns will **usually** have a solution set that’s a plane. (e.g., Example 2 or $3x + 2y - 5z = 1$)
* Two linear equations with 3 unknowns will **usually** have a solution set that’s a line. (e.g., Example 3 or $3x + 2y - 5z = 1$ and $x + z = 2$)
* Three linear equations with 3 unknowns will **usually** have a solution set that’s a point (i.e., a unique solution).

**Question**
When do these heuristic guidelines fail?

Vectors
We’ll study vectors both as formal mathematical objects and as tools for modelling the physical world.

**Definition**
A vector is an object that has both magnitude and direction.

Physical quantities such as velocity, force, momentum, torque, electromagnetic field strength are all “vector quantities” in that to specify them requires both a magnitude and a direction.
Vectors

Definition

A vector is an object that has both magnitude and direction.

We represent vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by arrows. For example, the vector $\mathbf{v}$ has initial point $A$ and terminal point $B$ and we write $\mathbf{v} = \vec{AB}$.

The zero vector $\mathbf{0}$ has length zero (and no direction).

Since a vector doesn’t have “location” as one of its properties, we can slide the arrow around as long as we don’t rotate or stretch it.

We can describe a vector using the coordinates of its head when its tail is at the origin, and we call these the components of the vector. Thus in this example $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and we say the components of $\mathbf{v}$ are 1 and 2.

Vector Addition

If an arrow representing $\mathbf{v}$ is placed with its tail at the head of an arrow representing $\mathbf{u}$, then an arrow from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$ represents the sum $\mathbf{u} + \mathbf{v}$.

Suppose that $\mathbf{u}$ has components $a$ and $b$ and that $\mathbf{v}$ has components $x$ and $y$. Then $\mathbf{u} + \mathbf{v}$ has components $a + x$ and $b + y$:
Scalar Multiplication

If \( \mathbf{v} \) is a vector, and \( t \) is a real number (scalar), then the scalar multiple of \( \mathbf{v} \) is a vector with magnitude \( |t| \) times that of \( \mathbf{v} \), and direction the same as \( \mathbf{v} \) if \( t > 0 \), or opposite to that of \( \mathbf{v} \) if \( t < 0 \).

If \( t = 0 \), then \( t \mathbf{v} \) is the zero vector \( \mathbf{0} \).

If \( \mathbf{u} \) has components \( a \) and \( b \), then \( t \mathbf{v} \) has components \( tx \) and \( ty \):

\[
t \mathbf{v} = t (x, y) = (tx, ty).
\]

Example

Example 4

A river flows north at 1km/hr, and a swimmer moves at 2km/hr relative to the water.

- At what angle to the bank must the swimmer move to swim east across the river?
- What is the speed of the swimmer relative to the land?

There are several velocities to be considered:
The velocity of the river, \( \mathbf{F} \), with \( ||\mathbf{F}|| = 1 \);
The velocity of the swimmer relative to the water, \( \mathbf{S} \), so that \( ||\mathbf{S}|| = 2 \);
The resultant velocity of the swimmer, \( \mathbf{F} + \mathbf{S} \), which is to be perpendicular to \( \mathbf{F} \).

The problem is to determine the direction of \( \mathbf{S} \) and the magnitude of \( \mathbf{F} + \mathbf{S} \).

From the figure it follows that the angle between \( \mathbf{S} \) and \( \mathbf{F} \) must be \( \frac{2\pi}{3} \) and the resulting speed will be \( \sqrt{3} \) km/hour.
Standard basis vectors in \( \mathbb{R}^2 \)

The vector \( \mathbf{i} \) has components 1 and 0, and the vector \( \mathbf{j} \) has components 0 and 1.

\[
\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

The vector \( \mathbf{r} \) from the origin to the point \((x, y)\) has components \(x\) and \(y\) and can be expressed in the form

\[
\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = x \mathbf{i} + y \mathbf{j}.
\]

The length of a vector \( \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \) is given by

\[
\|\mathbf{v}\| = \sqrt{x^2 + y^2}.
\]

Standard basis vectors in \( \mathbb{R}^3 \)

In the Cartesian coordinate system in 3-space we define three standard basis vectors \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) represented by arrows from the origin to the points \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\) respectively:

\[
\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Any vector can be written as a sum of scalar multiples of the standard basis vectors:

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \mathbf{i} + b \mathbf{j} + c \mathbf{k}.
\]

If \( \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \), the length of \( \mathbf{v} \) is defined as

\[
\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}.
\]

This is just the distance from the origin (with coordinates 0, 0, 0) of the point with coordinates \(a, b, c\).

A vector with length 1 is called a unit vector.

If \( \mathbf{v} \) is not zero, then \( \frac{\mathbf{v}}{\|\mathbf{v}\|} \) is the unit vector in the same direction as \( \mathbf{v} \).

The zero vector is not given a direction.
Example 5
The midpoints of the four sides of any quadrilateral are the vertices of a parallelogram.

Can you prove this using vectors?

Hint: how can you tell if two vectors are parallel? How can you tell if they have the same length?

Example 6
A boat travels due north to a marker, then due east, as shown:

Travelling at a speed of 10 knots with respect to the water, the boat must head 30° west of north on the first leg because of the water current. After rounding the marker and reducing speed to 5 knots with respect to the water, the boat must be steered 60° south of east to allow for the current.

Determine the velocity $u$ of the water current (assumed constant).

A diagram is helpful. The vector $u$ represents the velocity of the river current, and has the same magnitude and direction in both diagrams.

Applying the sine rule, we have

\[
\frac{\sin \theta}{10} = \frac{\sin \frac{\pi}{6}}{||u||} \quad \text{and} \quad \frac{\cos \theta}{5} = \frac{\sin \frac{\pi}{3}}{||u||}
\]

which are easily solvable for $||u||$ and $\theta$, and hence give $u$.\[\square\]
Example 7
An aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northwest at 100 km/h?

Solution The problem is 2-dimensional, so we can use plane vectors. Choose a coordinate system so that the $x$- and $y$-axes point east and north respectively.

We want $\vec{v}_{aircraft \ rel \ ground}$ to be in an easterly direction, that is, in the positive direction of the $x$-axis. So for ground speed of the aircraft $v$, we have

$$\vec{OQ} = v_{air rel ground}$$

$$= 100 \cos(-\pi/4)i + 100 \sin(-\pi/4)j$$

$$= 50\sqrt{2}i - 50\sqrt{2}j$$

$$\vec{OP} = v_{aircraft \ rel \ air}$$

$$= 750 \cos \theta i + 750 \sin \theta j$$

$$\vec{OR} = v_{aircraft \ rel \ ground}$$

$$= \vec{OP} + \vec{OQ}$$

$$= (750 \cos \theta i + 750 \sin \theta j) + (50\sqrt{2}i - 50\sqrt{2}j)$$

$$= (750 \cos \theta + 50\sqrt{2})i + (750 \sin \theta - 50\sqrt{2})j$$

Comparing the two expressions for $\vec{OR}$ we get

$$vi = (750 \cos \theta + 50\sqrt{2})i + (750 \sin \theta - 50\sqrt{2})j.$$ 

This implies that

$$750 \sin \theta - 50\sqrt{2} = 0 \quad \leftrightarrow \quad \sin \theta = \frac{\sqrt{2}}{15}.$$ 

This gives $\theta \approx 0.1$ radians $\approx 5.4^\circ$.

Using this information $v$ can be calculated, as well as the time to travel a given distance.